MATH2010E TUTO11

Thm (Lagrange Multipliers)

Let
- $f, g : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subset \mathbb{R}^n$ open)
- $S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$ be a level set of g

Suppose
- $\vec{a} \in S$ is a local extremum of f restricted to S (i.e. under the constraint $g = c$)
- $\vec{\nabla} g(\vec{a}) \neq \vec{0}$

Then
- $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$
- $g(\vec{a}) = c$

where λ is called a Lagrange Multiplier

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Thm (Lagrange Multipliers)

Let
- $f, g_1, g_2 : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subset \mathbb{R}^n$ open)
- $S = g^{-1}(c) = \{x \in \Omega : g_1(x) = c_1\}$ be a level set of g
  $g_2(x) = c_2$

Suppose
- $\vec{a} \in S$ is a local extremum of f restricted to S
  (i.e. under the constraint $g = c$)
- $\cancel{\vec{\nabla} g(\vec{a}) \neq \vec{0}}$ $\nabla g_1(a)$, $\nabla g_2(a)$ are linearly independent

Then
- $\vec{\nabla} f(\vec{a}) = \cancel{\lambda \vec{\nabla} g(\vec{a})}$ ~~for some~~ $\cancel{\lambda \in \mathbb{R}}$
  $\lambda_1 \nabla g_1(a) + \lambda_2 \nabla g_2(a)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$
- $g_1(\vec{a}) = c_1$, $g_2(a) = c_2$

where λ is called a Lagrange Multiplier

11. Rectangle of greatest area in an ellipse Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.

Ans: Maximize $f(x,y) := (2x)(2y) = 4xy$

under constraint $g(x,y) := x^2/16 + y^2/9 - 1 = 0$

both C^1

$\nabla f = (4y, 4x)$, $\nabla g = \left(\frac{x}{8}, \frac{2y}{9}\right) \neq \vec{0}$ on S

Clearly $S := g^{-1}(0)$ is closed and bounded

By EVT, f has a max on S.

By Lagrange Multiplier, a max pt. satisfies

$$\begin{cases} \nabla f = \lambda \nabla g & \text{for some } \lambda \in \mathbb{R} \\ g = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 4y = \lambda \cdot \frac{x}{8} & (1) \\ 4x = \lambda \cdot \frac{2y}{9} & (2) \\ \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0 & (3) \end{cases}$$

(1) : $\lambda = \frac{32y}{x}$ (note $x, y, \lambda \neq 0$)

(2) : $4x = \left(\frac{32y}{x}\right)\left(\frac{2y}{9}\right)$

$x^2 = \frac{16}{9}y^2$ $\quad x = \pm \frac{4}{3}y$

(3): $\frac{2y^2}{9} = 1$ $\quad y = \pm \frac{3}{\sqrt{2}}$

So the critical pts are $\left(2\sqrt{2}, \frac{3}{\sqrt{2}}\right)$, $\left(2\sqrt{2}, \frac{-3}{\sqrt{2}}\right)$, $\left(-2\sqrt{2}, \frac{3}{\sqrt{2}}\right)$, $\left(-2\sqrt{2}, \frac{3}{\sqrt{2}}\right)$.

Comparing values of f at these pts :

$f\left(2\sqrt{2}, \frac{3}{\sqrt{2}}\right) = f\left(-2\sqrt{2}, -\frac{3}{\sqrt{2}}\right) = 24$, $\quad f\left(2\sqrt{2}, -\frac{3}{\sqrt{2}}\right) = f\left(-2\sqrt{2}, \frac{3}{\sqrt{2}}\right) = -24$.

So the greatest area is 24, when width $= 4\sqrt{2}$, height $= \frac{6}{\sqrt{2}}$

43. Extrema on a circle of intersection Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.

closed and bdd $\Rightarrow$ extrema do exist

Ans :

$$\begin{cases} f(x,y,z) = xy + z^2 \\ g_1(x,y,z) := y - x \\ g_2(x,y,z) := x^2+y^2+z^2-4. \end{cases}$$

Consider

$$F(x,y,z,\lambda) := f(x,y,z) - \lambda_1 g_1(x,y,z) - \lambda_2 g_2(x,y,z)$$
$$= xy + z^2 - \lambda_1(y-x) - \lambda_2(x^2+y^2+z^2-4)$$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = y + \lambda_1 - 2\lambda_2 x & (1) \\ 0 = \frac{\partial F}{\partial y} = x - \lambda_1 - 2\lambda_2 y & (2) \\ 0 = \frac{\partial F}{\partial z} = 2z - 2\lambda_2 z & (3) \\ 0 = \frac{\partial F}{\partial \lambda_1} = -(y-x) & (4) \\ 0 = \frac{\partial F}{\partial \lambda_2} = -(x^2+y^2+z^2-4) & (5) \end{cases}$$

(3): $2z(1-\lambda_2) = 0 \Rightarrow z = 0$ or $\lambda_2 = 1$

Case 1 : $z = 0$.

(4), (5) : $2x^2 = 4 \Rightarrow x = \pm\sqrt{2} = y$

Case 2 : $\lambda_2 = 1$

(1), (2) : $\begin{cases} y + \lambda_1 - 2x = 0 \\ x - \lambda_1 - 2y = 0 \end{cases} \Rightarrow -(x+y) = 0 \Rightarrow x = y = 0$

(5) : $z^2 = 4 \Rightarrow z = \pm 2$

Critical pts : $(\sqrt{2}, \sqrt{2}, 0)$, $(-\sqrt{2}, -\sqrt{2}, 0)$, $(0, 0, \pm 2)$

Comparing values : $f(\sqrt{2}, \sqrt{2}, 0) = f(-\sqrt{2}, -\sqrt{2}, 0) = 2 \leftarrow$ min (on $g_1^{-1}(0) \cap g_2^{-1}(0)$)

$f(0,0,\pm 2) = 4 \leftarrow$ max (on $g_1^{-1}(0) \cap g_2^{-1}(0)$)

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45. The condition $\nabla f = \lambda \nabla g$ is not sufficient Although $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x, y)$ subject to the conditions $g(x, y) = 0$ and $\nabla g \neq \mathbf{0}$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x, y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4, 4)$ and $(-4, -4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.

Ans: Maximize $f(x,y) = x + y$

with constraint $g(x,y) = xy = 16$

Then $\nabla f = (1, 1)$

$\nabla g = (y, x) \neq \vec{0}$ if $(x,y) \neq (0,0)$

Solving $\begin{cases} \nabla f = \lambda \nabla g \\ g = 16 \end{cases}$

we have $\lambda = 1/4$, $(x,y) = (4,4)$

or $\lambda = -1/4$, $(x,y) = (-4,-4)$.

However $x+y$ has no max or min on $xy = 16$

because $x + \frac{16}{x} \to \infty$ if $x \to \infty$

$\to -\infty$ if $x \to -\infty$

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47. a. Maximum on a sphere Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centered at the origin of a Cartesian abc-coordinate system is $(r^2/3)^3$.

b. Geometric and arithmetic means Using part (a), show that for nonnegative numbers a, b, and c,

$$(abc)^{1/3} \le \frac{a+b+c}{3};$$

that is, the *geometric mean* of three nonnegative numbers is less than or equal to their *arithmetic mean*.

Ans: a) Maximize $f(a,b,c) = a^2b^2c^2$

under constraint $g(a,b,c) = a^2+b^2+c^2 = r^2$

Then $\nabla f = (2ab^2c^2, 2a^2bc^2, 2a^2b^2c)$

$\nabla g = (2a, 2b, 2c) \neq \vec{0}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = r^2 \end{cases} \iff \begin{cases} 2ab^2c^2 = 2a\lambda & (1) \\ 2a^2bc^2 = 2b\lambda & (2) \\ 2a^2b^2c = 2c\lambda & (3) \\ a^2+b^2+c^2 = r^2 & (4) \end{cases}$$

(1) $\times a$, (2) $\times b$, (3) $\times c$:

$$a^2b^2c^2 = a^2\lambda = b^2\lambda = c^2\lambda$$

Case 1 : $\lambda = 0$

$\Rightarrow f(a,b,c) = a^2b^2c^2 = 0 \leftarrow$ min (on $a^2+b^2+c^2=r^2$)

Case 2 : $\lambda \neq 0$

$\Rightarrow a, b, c \neq 0$

$\Rightarrow a^2 = b^2 = c^2 \left(= \frac{a^2b^2c^2}{\lambda} \right)$

$\overset{(4)}{\Rightarrow} a^2 = b^2 = c^2 = \frac{r^2}{3}$

$\Rightarrow f(a,b,c) = \left(\frac{r^2}{3}\right)^3 \leftarrow$ max (on $a^2+b^2+c^2=r^2$)

b) Take $r^2 = a+b+c$

Then $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ lies on the sphere of radius r centred at $\vec{0}$.

By a),
$$(\sqrt{a})^2(\sqrt{b})^2(\sqrt{c})^2 \leq \left(\frac{r^2}{3}\right)^3 = \left(\frac{a+b+c}{3}\right)^3$$

i.e.
$$(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3}$$

//

Example Transform the following quadratic constraints to the standard form.

a) $x^2 + xy + y^2 = 9$ ①

b) $xy - yz - zx = 3$

Ans:

$$x^2 + xy + y^2 = 9$$

$$(x+y)^2 - xy = 9$$

$$(x+y)^2 - \frac{1}{4}\left[(x+y)^2 - (x-y)^2\right] = 9$$

Note $4xy = (x+y)^2 - (x-y)^2$

$$\frac{3(x+y)^2}{4} + \frac{(x-y)^2}{4} = 9$$

$$\frac{(x+y)^2}{12} + \frac{(x-y)^2}{36} = 1 \quad (*)$$

(the calculation is incorrect in tutorial)

Put $\begin{cases} u = \frac{x+y}{\sqrt{2}} \\ v = \frac{x-y}{\sqrt{2}} \end{cases}$

Then $(*)$ becomes

$$\frac{u^2}{(\sqrt{6})^2} + \frac{v^2}{(\sqrt{18})^2} = 1$$

an ellipse //

Example Transform the following quadratic constraints to the standard form.

a) $x^2 + xy + y^2 = 9$

b) $xy - yz - zx = 3$

Ans: b) $xy - yz - zx = 3$

$$\frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 - (x+y)z = 3 \qquad (*)$$

I made a mistake here in tutorial

Let $\begin{cases} u = x+y \\ v = x-y \end{cases}$

$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\sqrt{\frac{5}{3}}\right.$

Then $(*)$ becomes

$$\frac{1}{4}u^2 - \frac{1}{4}v^2 - uz = 3$$

$$\frac{1}{4}(u-2z)^2 - \frac{1}{4}v^2 - z^2 = 3 \qquad (**)$$

Let $\begin{cases} u' = \frac{1}{2}(u-2z) = \frac{1}{2}(x+y-2z) \\ v' = \frac{1}{2}v = \frac{1}{2}(x-y) \\ w' = z \end{cases}$

Then $(**)$ becomes

$$(u')^2 - (v')^2 - (w')^2 = 3,$$ a hyperboloid (of 2 sheets)

//

or

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu. \tag{5}$$

The scalar equations in Equations (5) yield

$$\begin{aligned} 2x &= 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z, \\ 2y &= 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z. \end{aligned} \tag{6}$$

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Equations (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points (1, 0, 0) and (0, 1, 0). This makes sense when you look at Figure 14.59.

If $x = y$, then Equations (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

Here we need to be careful, however. Although P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0). The point on the ellipse farthest from the origin is P_2. (See Figure 14.59.) ■

Exercises 14.8

Two Independent Variables with One Constraint

1. **Extrema on an ellipse** Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

2. **Extrema on a circle** Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

3. **Maximum on a line** Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

4. **Extrema on a line** Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.

5. **Constrained minimum** Find the points on the curve $xy^2 = 54$ nearest the origin.

6. **Constrained minimum** Find the points on the curve $x^2y = 2$ nearest the origin.

7. Use the method of Lagrange multipliers to find

   a. **Minimum on a hyperbola** The minimum value of $x + y$, subject to the constraints $xy = 16$, $x > 0$, $y > 0$

   b. **Maximum on a line** The maximum value of xy, subject to the constraint $x + y = 16$.

   Comment on the geometry of each solution.

8. **Extrema on a curve** Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

9. **Minimum surface area with fixed volume** Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 16π cm^3.

10. **Cylinder in a sphere** Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What *is* the largest surface area?

11. **Rectangle of greatest area in an ellipse** Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.

12. **Rectangle of longest perimeter in an ellipse** Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What *is* the largest perimeter?

13. **Extrema on a circle** Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

14. **Extrema on a circle** Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.

15. **Ant on a metal plate** The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

16. **Cheapest storage tank** Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Three Independent Variables with One Constraint

17. **Minimum distance to a point** Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

18. **Maximum distance to a point** Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$.

19. **Minimum distance to the origin** Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin.

20. **Minimum distance to the origin** Find the point on the surface $z = xy + 1$ nearest the origin.

21. **Minimum distance to the origin** Find the points on the surface $z^2 = xy + 4$ closest to the origin.

22. **Minimum distance to the origin** Find the point(s) on the surface $xyz = 1$ closest to the origin.

23. **Extrema on a sphere** Find the maximum and minimum values of

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

24. **Extrema on a sphere** Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

25. **Minimizing a sum of squares** Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

26. **Maximizing a product** Find the largest product the positive numbers x, y, and z can have if $x + y + z^2 = 16$.

27. **Rectangular box of largest volume in a sphere** Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

28. **Box with vertex on a plane** Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $x/a + y/b + z/c = 1$, where $a > 0$, $b > 0$, and $c > 0$.

29. **Hottest point on a space probe** A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

30. **Extreme temperatures on a sphere** Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.

31. **Cobb-Douglas production function** During the 1920s, Charles Cobb and Paul Douglas modeled total production output P (of a firm, industry, or entire economy) as a function of labor hours involved x and capital invested y (which includes the monetary worth of all buildings and equipment). The Cobb-Douglas production function is given by

$$P(x, y) = kx^{\alpha}y^{1-\alpha},$$

where k and α are constants representative of a particular firm or economy.

   a. Show that a doubling of both labor and capital results in a doubling of production P.

   b. Suppose a particular firm has the production function for $k = 120$ and $\alpha = 3/4$. Assume that each unit of labor costs \$250 and each unit of capital costs \$400, and that the total expenses for all costs cannot exceed \$100,000. Find the maximum production level for the firm.

32. (*Continuation of Exercise 31.*) If the cost of a unit of labor is c_1 and the cost of a unit of capital is c_2, and if the firm can spend only B dollars as its total budget, then production P is constrained by $c_1x + c_2y = B$. Show that the maximum production level subject to the constraint occurs at the point

$$x = \frac{\alpha B}{c_1} \quad \text{and} \quad y = \frac{(1 - \alpha)B}{c_2}.$$

33. **Maximizing a utility function: an example from economics** In economics, the usefulness or *utility* of amounts x and y of two capital goods G_1 and G_2 is sometimes measured by a function $U(x, y)$. For example, G_1 and G_2 might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If G_1 costs a dollars per kilogram, G_2 costs b dollars per kilogram, and the total amount allocated for the purchase of G_1 and G_2 together is c dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

    Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$2x + y = 30.$$

Find the maximum value of U and the corresponding values of x and y subject to this latter constraint.

34. **Blood types** Human blood types are classified by three gene forms A, B, and O. Blood types AA, BB, and OO are *homozygous*, and blood types AB, AO, and BO are *heterozygous.* If p, q, and r represent the proportions of the three gene forms to the population, respectively, then the *Hardy-Weinberg Law* asserts that the proportion Q of heterozygous persons in any specific population is modeled by

$$Q(p, q, r) = 2(pq + pr + qr),$$

subject to $p + q + r = 1$. Find the maximum value of Q.

35. **Length of a beam** In Section 4.6, Exercise 39, we posed a problem of finding the length L of the shortest beam that can reach over a wall of height h to a tall building located k units from the wall. Use Lagrange multipliers to show that

$$L = (h^{2/3} + k^{2/3})^{3/2}.$$

36. **Locating a radio telescope** You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical, with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?

Extreme Values Subject to Two Constraints

37. Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

38. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

39. Minimum distance to the origin Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.

40. Maximum value on line of intersection Find the maximum value that $f(x, y, z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.

41. Extrema on a curve of intersection Find the extreme values of $f(x, y, z) = x^2yz + 1$ on the intersection of the plane $z = 1$ with the sphere $x^2 + y^2 + z^2 = 10$.

42. a. Maximum on line of intersection Find the maximum value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$.

b. Give a geometric argument to support your claim that you have found a maximum, and not a minimum, value of w.

43. Extrema on a circle of intersection Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.

44. Minimum distance to the origin Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

Theory and Examples

45. The condition $\nabla f = \lambda \nabla g$ is not sufficient Although $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x, y)$ subject to the conditions $g(x, y) = 0$ and $\nabla g \neq \mathbf{0}$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x, y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4, 4)$ and $(-4, -4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.

46. A least squares plane The plane $z = Ax + By + C$ is to be "fitted" to the following points (x_k, y_k, z_k):

$$(0, 0, 0), \qquad (0, 1, 1), \qquad (1, 1, 1), \qquad (1, 0, -1).$$

Find the values of A, B, and C that minimize

$$\sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2,$$

the sum of the squares of the deviations.

47. a. Maximum on a sphere Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centered at the origin of a Cartesian abc-coordinate system is $(r^2/3)^3$.

b. Geometric and arithmetic means Using part (a), show that for nonnegative numbers a, b, and c,

$$(abc)^{1/3} \leq \frac{a + b + c}{3};$$

that is, the *geometric mean* of three nonnegative numbers is less than or equal to their *arithmetic mean*.

48. Sum of products Let $a_1, a_2, \dots, a_n$ be n positive numbers. Find the maximum of $\sum_{i=1}^{n} a_i x_i$ subject to the constraint $\sum_{i=1}^{n} x_i^2 = 1$.

COMPUTER EXPLORATIONS

In Exercises 49–54, use a CAS to perform the following steps implementing the method of Lagrange multipliers for finding constrained extrema:

a. Form the function $h = f - \lambda_1 g_1 - \lambda_2 g_2$, where f is the function to optimize subject to the constraints $g_1 = 0$ and $g_2 = 0$.

b. Determine all the first partial derivatives of h, including the partials with respect to λ_1 and λ_2, and set them equal to 0.

c. Solve the system of equations found in part (b) for all the unknowns, including λ_1 and λ_2.

d. Evaluate f at each of the solution points found in part (c) and select the extreme value subject to the constraints asked for in the exercise.

49. Minimize $f(x, y, z) = xy + yz$ subject to the constraints $x^2 + y^2 - 2 = 0$ and $x^2 + z^2 - 2 = 0$.

50. Minimize $f(x, y, z) = xyz$ subject to the constraints $x^2 + y^2 - 1 = 0$ and $x - z = 0$.

51. Maximize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $2y + 4z - 5 = 0$ and $4x^2 + 4y^2 - z^2 = 0$.

52. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x^2 - xy + y^2 - z^2 - 1 = 0$ and $x^2 + y^2 - 1 = 0$.

53. Minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to the constraints $2x - y + z - w - 1 = 0$ and $x + y - z + w - 1 = 0$.

54. Determine the distance from the line $y = x + 1$ to the parabola $y^2 = x$. (*Hint:* Let (x, y) be a point on the line and (w, z) a point on the parabola. You want to minimize $(x - w)^2 + (y - z)^2$.)

14.9 Taylor's Formula for Two Variables

In this section we use Taylor's formula to derive the Second Derivative Test for local extreme values (Section 14.7) and the error formula for linearizations of functions of two independent variables (Section 14.6). The use of Taylor's formula in these derivations leads to an extension of the formula that provides polynomial approximations of all orders for functions of two independent variables.

Derivation of the Second Derivative Test

Let $f(x, y)$ have continuous partial derivatives in an open region R containing a point $P(a, b)$ where $f_x = f_y = 0$ (Figure 14.60). Let h and k be increments small enough to put the

Thm (**Lagrange Multipliers**)

Let
- $f, g : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subset \mathbb{R}^n$ open)
- $S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$ be a level set of g

Suppose
- $\vec{a} \in S$ is a local extremum of f restricted to S
  (i.e. under the constraint $g = c$)
- $\vec{\nabla} g(\vec{a}) \neq \vec{0}$

Then
- $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$
- $g(\vec{a}) = c$

where λ is called a Lagrange Multiplier

Reduction to unconstrainted problem (By Lagrange Multiplier)

Finding extrema of $f(\vec{x})$ with constraint $g(\vec{x}) = c$

$$\Updownarrow$$

Finding extrema of $F(\vec{x}, \lambda) = f(\vec{x}) - \lambda(g(\vec{x}) - c)$

without constraint

(but more variables : adding λ as a new variable)

Idea: $F(\vec{x}, \lambda) = F(x_1, \cdots, x_n, \lambda)$ is $n+1$ variables